SMALL-AMPLITUDE CONVECTIVE MAGNETOENTROPIC WAVES

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UDC 532.511:538.6

The propagation of small perturbations of magnetic field, pressure, density, and entropy in an ideal conducting medium located in a constant and uniform magnetic field $\vec{H}_0$ is investigated, taking account of the effect of gravity. It is shown that there exist convective magnetoentropic waves differing both from internal and from magnetohydrodynamic waves. The characteristics and conditions of propagation of these waves in an ideal fluid are investigated.

In magnetohydrodynamics [1] there exists the concept of the "freezing in" of magnetic lines of force. It signifies parallelism of the changes of the vector of the magnetic field strength and an element of length of the "fluid line."

As seen from the heat-transfer equation, which in the case of an ideal fluid reduces to the equation of the conservation of entropy

$$\frac{\partial S}{\partial t} + \vec{v} \cdot \nabla S = 0, \tag{1}$$

the isentropic lines are also "frozen in." "Freezing in" of entropy signifies the dependenece of the change of entropy with time on the velocity of the fluid, i.e., it is understood in the same sense as "freezing in" of magnetic lines of force. Owing to "freezing in" of the magnetic lines of force and the isentropic lines, induced oscillatory motion in a medium should propagate as waves of the magnetic field, pressure, velocity, density, and entropy.

This statement of the problem is described by a system of magnetohydrodynamic equations under the following conditions.

In a constant and uniform homogeneous field $\vec{H}_0$ there is a conducting fluid having viscosity, electrical resistance, and thermal conductivity so small that the effect of energy dissipation associated with these values on the propagation of perturbations may be neglected. In an equilibrium state there exists a constant entropy gradient directed along the gravitational field in which the conducting fluid is located. The gravitational field is uniform and constant in time.

The solution of the problem stated here is determined by the following system of equations:

$$\operatorname{div} H = 0, \tag{2}$$

$$\frac{\partial \vec{H}}{\partial t} = \operatorname{rot}\,[\vec{v}\vec{H}], \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0, \tag{4}$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v}\nabla)\vec{v} = -\nabla p + \frac{1}{4\pi} \operatorname{rot} \vec{H} \times \vec{H} + \rho \vec{g}, \tag{5}$$

$$\frac{\partial S}{\partial t} + \nabla S \cdot \vec{v} = 0. \tag{6}$$

Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 19, No. 4, pp. 715-721, October, 1970. Original article submitted July 16, 1969.

In addition to these equations, the solution of the problem requires consideration of the equation of state of matter, relating ρ, p, and S. In a general form it can be written:

$$f(\rho,\ p,\ S) = 0. \tag{7}$$

In a perturbed state the quantities characterizing the medium and magnetic field have the following form:

$$\vec{H} = \vec{H}_0 + \vec{h},\ p = p_0 + p',\ \rho = \rho_0 + \rho',\ S = S_0 + S',$$

where the subscript "0" denotes constant equilibrium values of the quantities and the prime their perturbation.

We will seek waves traveling along the x axis in the presence of an entropy gradient and gravitational field directed along the y axis. In a gravitational field at a given entropy gradient, ρ_0 should, generally speaking, depend on y; to allow ρ_0 to be considered constant, we will consider a fluid layer of small thickness. We will also consider that in an equlibrium state the fluid is at rest, i.e., $\vec{v}_0 = 0$, and the perturbations of $\vec{h}$, p', ρ', S', and $\vec{v}$ are of the same order of smallness.

The following relationship is obtained from the solution of system (2)-(6) in the zeroth approximation:

$$-\nabla p_0 = \rho_0 \vec{g}. \tag{8}$$

Since we will seek the solution of system (2)-(7) in a linear approximation, we will take as the equation of state the expression

$$p = \alpha\rho + \beta S, \tag{9}$$

where α is the square of the sound velocity, and β is the coefficient of entropic compressibility; p in the equation of state is understood as the sum of the hydrostatic and induced magnetic pressures.

The linearized system of Eqs. (2)-(7) for perturbations with consideration of the comments made above concerning the pressure has the form

$$\operatorname{div} \vec{h} = 0, \tag{10}$$

$$\frac{\partial \vec{h}}{\partial t} = (H_0 \nabla)\vec{v} - H_0(\nabla \vec{v}), \tag{11}$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \vec{v} = 0, \tag{12}$$

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\alpha\nabla\rho' - \beta\nabla S' + \frac{1}{4\pi}(H_0\nabla)\, h + \rho'\vec{g}, \tag{13}$$

$$\frac{\partial S'}{\partial t} + \nabla S_0 \cdot \vec{v} = 0. \tag{14}$$

The solution of system (10)-(14) is sought in the form of plane waves with complex amplitudes

$$\begin{aligned} \vec{h} &= \vec{h}_a \exp i\,\alpha_h \exp i(\vec{k}\vec{r} - \omega t), \\ \vec{v} &= \vec{v}_a \exp i\alpha_v \exp i(\vec{k}\vec{r} - \omega t), \\ S' &= S'_a \exp i\,\alpha_s \exp i(\vec{k}\vec{r} - \omega t), \\ \rho' &= \rho'_a \exp i\,\alpha_\rho \exp i(\vec{k}\vec{r} - \omega t). \end{aligned} \tag{15}$$

Substitution of (15) into (10)-(14) gives the following system of linear algebraic equations:

$$-Uh_z = H_{0x}v_z,\quad Uv_z = (H_{0x}/4\pi\rho_0)\cdot h_z, \tag{16}$$

$$Uh_y = H_{0x}v_y + H_{0y}v_x, \tag{17}$$

$$Uv_x = (\alpha/U)\, v_x - (\beta\gamma/Uk\rho_0)\, iv_y, \tag{18}$$

$$Uv_y = -(H_{0x}/4\pi\rho_0)\, h_y + (gi/Uk)\, v_x, \tag{19}$$

$$\rho' = (\rho_0/U)\, v_x, \tag{20}$$

$$S' = -(i\gamma/\omega)\, v_y, \tag{21}$$

$$\vec{h} \perp \vec{k},\ h_x = 0, \tag{22}$$

where $h_z = \underset{a}{h_z} \exp i\alpha_h$, etc., and $U = \omega/k$ is the phase velocity. Obviously the system breaks down into two subsystems which describe different groups of waves: in one only the components h_z and v_z oscillate and in the other, v_x, v_y, h_y, ρ', and S'. As was initially suggested, entropic waves exist. From the compatibility condition (16) we obtain the phase velocity for the first group of waves

$$U = \pm H_{0x}/\sqrt{4\pi\rho_0},$$

i.e., the oscillations of h_z and v_z are Alfven [2].

The zero equality of the determinant of the second subsystem gives the following dispersion equation:

$$(U^2 - \alpha)(U^2 - H_{0x}^2/4\pi\rho_0) = (\beta\gamma g/\rho_0\omega^2)\, U^2 - \frac{H_{0x}H_{0y}}{\omega\rho_0^2}\beta\gamma i U. \tag{23}$$

Equation (23) represents a quartic equation in U, becoming quartic when $H_{0y} = 0$. The contribution of the term with H_{0y} exists only in the presence of an entropy gradient. We see from Eq. (23) that the propagation of the second group of waves is characterized by the presence of dispersion.

Dispersion equation (23) is obtained on the assumption that in the equation of state the pressure represents the sum of the hydrostatic and magnetic pressures, i.e.,

$$p = p_{\text{hydr}} + H^2/8\pi.$$

If in the equation of state we understand by pressure only the hydrostatic pressure, as is usually done, the dispersion equation for the second group of waves has now a different form:

$$(U^2 - \alpha)(U^2 - H_{0x}^2/4\pi\rho_0) = (H_{0y}^2/4\pi\rho_0 + \beta\gamma g/\rho_0\omega^2)\, U^2 + (\beta\gamma/\rho_0\omega - g/4\pi\rho_0\omega)\, H_{0x}H_{0y} i U. \tag{24}$$

We see from Eqs. (23) and (24) that in the case

$$1)\ p = p_{\text{hydr}} + H^2/8\pi \tag{25}$$

the dispersion is due only to the entropy gradient and the gravitational field has no effect on the existence of dispersion;

$$2)\ p = p_{\text{hydr}}$$

dispersion is observed and in the absence of the entropy gradient, it can be due only to the gravitational field.

If the field H_0 is directed along the x axis, both cases coincide, and then, the dispersion equation has the following form:

$$(U^2 - \alpha)(U^2 - H_{0x}^2/4\pi\rho_0) = (\beta\gamma g/\rho_0\omega^2)\, U^2. \tag{26}$$

In this case dispersion is observed only with the simultaneous existence of both the gravitational field and entropy gradient.

There figures in Eqs. (23) and (24) a term with an imaginary unit. This signifies that there exists either an attenuation or an increment of waves, which in both cases disappear if we consider waves propagating along the magnetic field. If we consider waves propagating in the xy plane, then in the first case attenuation (or increment) disappears when there is no entropy gradient and in the second case when both the entropy gradient and gravitational field are absent.

It is especially important to take into account the term $H^2/8\pi$ in strong magnetic fields, and therefore we will henceforth consider that the pressure is determined according to Eq. (25).

Equation (23), when $H_{0y} = 0$, has the following solution:

$$U = \left\{\frac{1}{2}(\alpha + \beta\gamma g/\rho_0\omega^2 + H_{0x}^2/4\pi\rho_0) \pm \sqrt{\frac{1}{4}(\alpha + \beta\gamma g/\rho_0\omega^2 + H_{0x}^2/4\pi\rho_0)^2 - \alpha(H_{0x}^2/4\pi\rho_0)}\right\}^{1/2}. \tag{27}$$

The relationships between the amplitudes of the different components are obtained from system (17)-(22):

$$v_z = -\frac{1}{\sqrt{4\pi\rho_0}} h_z, \tag{28}$$

$$v_y = -\frac{U}{H} h_y, \tag{29}$$

$$v_x = \frac{\beta\gamma i}{H_{0x}\rho_0\omega(1-\alpha/U^2)} h_y, \tag{30}$$

$$\rho' = (\rho_0/U)\, v_x, \tag{31}$$

$$S' = \frac{\gamma i}{\omega} v_y. \tag{32}$$

As we see from this system, the existence of an entropy gradient gives rise to a phase shift between components v_x and v_y, S' and v_y, v_x and h_y by $\pi/2$.

The phase velocities (27) characterize the propagation of the two wave groups in which the components v_x, v_y, h_y, ρ', and S' oscillate.

These phase velocities have the following form in units of Alfven velocity:

$$U_1' = U_1/V_A = 1,$$
$$U_{2,3}' = \frac{1}{\sqrt{2}}\{(\zeta^2+\eta^2+1) \pm \sqrt{(\zeta^2+\eta^2+1)^2 - 4\zeta^2}\}^{1/2}, \tag{33}$$

where the parameters $\zeta = V_S/V_A$, $\eta = V_E/V_A$ are introduced.

Weak and strong magnetic and gravitational fields have two limiting cases characterized by corresponding values of the parameters introduced above.

1. Strong magnetic and weak gravitational fields correspond to $\zeta \to 0$ and $\eta \ll 1$. In this case $(U_2')^2 = 1$, $(U_3')^2 = 0$, i.e., the phase velocity is equal to the Alfven velocity $(U_2)^2 = V_A^2$. This limiting case represents ordinary magnetohydrodynamic waves.

2. Weak magnetic and strong gravitational fields correspond to $\eta^2 \gg 1$. In addition, we will consider $\xi^2 \ll 1$. In this case $(U_2')^2 = 0$, $(U_3')^2 = \eta^2$, i.e., the phase velocity of the third group of waves is equal to the Brunt–Vaisala velocity, and the phase velocity of the second wave group is equal to zero:

$$U_3^2 = \beta\gamma g/\rho_0\omega^2. \tag{34}$$

It is obvious that this limiting case gives waves which exist only when there is an entropy gradient in the gravitational field. They are characterized by dispersion. The group velocity of these waves has the following form:

$$v_{gr} = \frac{1}{2\sqrt{k}} (\beta\gamma g/\rho_0)^{1/4}, \tag{35}$$

$\vec{v}_{gr} \parallel x$, since $\vec{k} \parallel x$. Components of the following form propagate with this velocity:

$$v_y = -\frac{1}{H_0} V_E h_y, \tag{36}$$

$$v_x = \frac{V_E \omega i}{H_{0x} g} h_y, \tag{37}$$

$$\rho' = \frac{\rho_0 \omega i}{H_{0x} g} V_E h_y, \tag{38}$$

$$S' = \frac{\gamma i}{\omega H_{0x}} V_E h_y. \tag{39}$$

Thus, perturbations of the quantities v_x, v_y, h_y, v_z, h_z, ρ', p', and S' propagate as plane waves in strong gravitational fields in the presence of an entropy gradient.

The existence of plane undamped waves depends on the direction of the entropy gradient and direction of the gravitational field.

1. If $\beta\gamma g > 0$, waves can propagate with velocity (34) for any relations between ξ and η.

2. If $\beta\gamma g < 0$, the following relation must be fulfilled for the existence of waves:

$$(\zeta - 1)^2 > \eta^2, \text{ i.e., } \beta\gamma g < \frac{\omega^2}{4\pi}(\sqrt{4\pi\rho_0\alpha} - H_{0x})^2. \tag{40}$$

Thus, if the entropy gradient and gravitational field are directed in the same direction, plane undamped waves v_x, v_y, h_y, ρ', p', and S' propagate for any relations between $\beta\gamma g$ and H_0. If ∇S_0 and g are directed in opposite directions, such waves exist only under condition (40).

We need point out that the second limiting case exists only provided $\beta\gamma g > 0$.

The results presented above were obtained on the assumption that the equation of state is used in the form $p = \alpha\rho + \beta S$ or $\rho = 1/\alpha\, p - \beta/\alpha S$. If we take into account only entropic compressibility, i.e., we set $1/\alpha = 0$ in the last equation of state, and we consider $\beta/\alpha = -\beta_1$ to be a finite number, Equation (23) will change to the following equation:

$$(V_{E1}^2 + 1)U^2 + \frac{H_{0x}H_{0y}\omega}{\rho_0 g}V_{E1}^2 U + V_A^2 = 0. \tag{41}$$

It follows from this equation that even in the ideal case: if in the equation of state we mean by pressure the sum of the hydrostatic and induced magnetic pressures, the existing waves attenuate when the field $\vec{H}_0$ is not along the x axis but in the xy plane.

The wave vector in the presence of attenuation is complex: $k = k_1 + ik_2$, where k_2 characterizes both attenuation ($k_2 > 0$) and increase ($k_2 < 0$). From (41), considering that $U = \omega/k$, and $k = k_1 + ik_2$, we obtain the following values for k_2 and k_1:

$$k_2 = \frac{2\pi H_{0y}\beta_1\gamma}{H_{0x}\rho_0}, \tag{42}$$

from which we see that a decrease or increase of amplitudes depends on the direction of the entropy gradient. With ∇S_0 parallel to the y axis the waves attenuate and when ∇S_0 is antiparallel to the y axis they increase and

$$k_1 = \pm(2\sqrt{\pi}/H_{0x})\sqrt{\omega\rho_0^2 - \beta_1\gamma g - \pi\left(\frac{H_{0y}\beta_1\gamma}{\rho_0}\right)^2}. \tag{43}$$

The waves will be weakly damped when $k_2/k_1 \ll 1$, i.e., when $H_{0y} \to 0$. Such damped (increasing) waves can exist only when

$$H_{0y}^2 < \frac{\omega^2\rho_0 - \beta_1\gamma g}{\pi(\beta_1\gamma)^2}\rho_0^2. \tag{44}$$

In view of the fact that $H_{0y}^2 > 0$, the denominator of the fraction on the right is always greater than zero; this condition reduces to the condition $\omega^2\rho_0 > \beta_1\gamma g$. Consequently, if the magnetic field H_0 is not directed along the direction of wave propagation, the waves attenuate and they are possible only when $\beta_1\gamma g < \omega^2\rho_0$.

If we seek waves traveling along the field, the dispersion equation for them will have the form

$$(V_{E1} + 1)U^2 - V_A^2 = 0, \tag{45}$$

whence

$$U^2 = V_A^2/(V_{E1}^2 + 1). \tag{46}$$

The existence of this velocity means that due to entropic compressibility alone in a constant, uniform magnetic field, the direction of which coincides with the direction of wave propagation, there exist, in the presence of an entropy gradient of the gravitational field, undamped convective magnetoentropic waves propagating with a velocity given by (45).

NOTATION

$\vec{H}$ is the vector of magnetic field strength;
$\vec{v}$ is the vector of fluid velocity;
$\vec{g}$ is the gravitational acceleration;
γ is the entropy gradient in fluid;

S is the entropy of fluid;
ρ is the density of fluid;
p is the pressure of fluid;
β is the entropic compressibility of fluid;
$\vec{k}$ is the wave vector;
ω is the wave frequency;
U is the phase velocity of wave;
v_{gr} is the group velocity of wave;
α is the square of the sound velocity;
V_S is the sound velocity;
V_A is the propagation velocity of Alfven waves;
V_E is the Brunt–Vaisala velocity.

LITERATURE CITED

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